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Heat transfer in plane Couette flow of a rarefied gas

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SUMMARY
In this paper Shen's method is used for solving the problem of heat transfer in plane Couette flow of a rarefied gas. An approximate distribution function is assumed and the appropriate transfer equations, which exhibit the proper collision effect between the molecules, are used to determine the temperature jumps at the plates.

1. Introduction

The problem of heat transfer in a gas flow between parallel plates using the linearized Boltzmann equation with the BGK-model is not a new one. Recently, Bhatnagar and Srivastava [1], studied the problem of heat transfer in a plane Couette flow by using the method of moments for the half range distributions. They determined approximate solutions of the heat flux vector and the temperature jumps at the two plates.

Previously, Shen [2] proposed a relaxation typè procedure for the transition regime of rarefied gas flows, which provides a complete solution valid for all Knudsen numbers. He made use of Lees' [3] two-stream Maxwellian in his assumed form of the distribution function. Later on, Sarin [4] has also used Shen's approach in a slightly generalized form, for solving the problem of Couette flow.

In the present paper we use Shen's approach to study the problem of heat transfer in plane Couette flow. In this procedure we make use of the appropriate transfer equations so as to calculate the unknown functions introduced in the distribution function. Numerical results have then been obtained for the heat flux vector and temperature jumps at the two plates.

2. Analysis

Consider a gas between two parallel plates. As the problem is one of heat transfer, the density variation is neglected. The two plates, $y=0$ and $y=h$, are maintained at constant temperatures T_0 and T_h respectively and the upper plate is moving with a relative speed U in the x-direction. We assume a steady state in which all the quantities depend only on the y-coordinate measured perpendicular to the plates. Denoting the relative change in the distribution function by ϕ and the dimensionless molecular velocity vector (cf. [1]) by (v_x, v_y, v_z) and its length by v, we introduce the function $\psi(v_y, y)$, in the BGK-model of the Boltzmann equation, defined as

$$\psi(v_y, y) = \int_{v_x=-\infty}^{\infty} \int_{v_z=-\infty}^{\infty} (v^2 - \tfrac{3}{2}) \exp(-v_x^2 - v_z^2)\, \phi\, dv_x dv_z\,,$$

which satisfies the following integro-differential equation (*cf.* [1]):

$$v_y \frac{\partial\psi}{\partial y} + \lambda\psi = \frac{\lambda(4v_y^4 - 4v_y^2 + 5)}{6\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \psi(v_y, y) \exp(-v_y^2) dv_y\,, \tag{1}$$

where λ is the inverse of the mean free path of the gas.

The boundary conditions to be satisfied by $\psi(v_y, y)$ are:

$$\psi^+(v_y, 0) = 0\,, \quad \psi^-(v_y, h) = \pi\left[U^2 + \frac{3}{2}\frac{T_h}{T_0} - \frac{3}{2}\right]\left(\frac{T_0}{T_h}\right)^{\frac{1}{2}\star} = \psi_w\,, \tag{2}$$

where ψ^+ and ψ^- are the half-range distribution functions for $v_y > 0$ and $v_y < 0$ respectively.

Following [2], we assume that the approximate distribution function which exhibits the proper collision effect has the form

$$\psi = \psi' \exp\left[-\frac{\lambda(y-y')}{v_y}\right] + \psi_0\left\{1 - \exp\left[-\frac{\lambda(y-y')}{v_y}\right]\right\}$$

with ψ' indicating the distribution prescribed at the position ($y'=0$ and $y'=h$) of the molecules having velocity v_y and where the choice of distribution ψ_0 is rather arbitrary. For simplicity we choose the approximate distribution function to be of the form

$$\psi(v_y, y) = \{1 - \exp[-\lambda y/v_y]\}\sum_{n=0}^{\infty} v_y^n X_n^+(y)\,, \text{ for } v_y < 0 \tag{4a}$$

$$\psi(v_y, y) = \psi_w \exp\left[-\frac{\lambda(y-h)}{v_y}\right] + \left\{1 - \exp\left[-\frac{\lambda(y-h)}{v_y}\right]\right\}\sum_{n=0}^{\infty} v_y^n X_n^-(y)\,, \text{ for } v_y > 0 \tag{4b}$$

where ψ_w is defined in equation (2) and $X_n^\pm(y)$ are unknown functions of y. Taking $n=0$, the functions $X_0^+(y)$ and $X_0^-(y)$ are found by using the appropriate transfer equations. Once these quantities are determined, the temperature jumps and the heat flux vector can be calculated. Making use of the two transfer equations for 1 and v_y, these turn out to be

$$\langle v_y \rangle = -q_y = \text{constant} \tag{5}$$

$$\langle v_y^2 \rangle = -q_y(\lambda y) + C\,. \tag{6}$$

Evaluating $\langle v_y \rangle$ and $\langle v_y^2 \rangle$, we get

$$\langle v_y \rangle = -\frac{1}{\pi^{\frac{1}{2}}}\left[\psi_w J_1\{\lambda(y-h)\} + X_0^-(y)(\tfrac{1}{2} - J_1\{\lambda(y-h)\}) - X_0^+(y)(\tfrac{1}{2} - J_1\{y\lambda\})\right], \tag{7}$$

$$\langle v_y^2 \rangle = \frac{1}{\pi^{\frac{1}{2}}}\left[\psi_w J_2\{\lambda(y-h)\} + X_0^-(y)\left(\frac{\pi^{\frac{1}{2}}}{4} - J_2\{\lambda(y-h)\}\right) + X_0^+(y)\left(\frac{\pi^{\frac{1}{2}}}{4} - J_2\{y\lambda\}\right)\right]\,, \tag{8}$$

where the auxiliary function $J_n(\alpha)$ is given as

$$J_n(\alpha) = \int_0^\infty \exp\left(-x^2 - \frac{\alpha}{x}\right)x^n dx\,, \qquad (n = 0, 1, 2, \ldots)\,. \tag{9}$$

From equations (5)–(8), we get the final equations

$$-X_0^-(y)[\tfrac{1}{2} - A_1] + X_0^+(y)[\tfrac{1}{2} - B_1] = \pi^{\frac{1}{2}} F_1 \tag{10}$$

$$X_0^-(y)\left[\frac{\pi^{\frac{1}{2}}}{4} - A_2\right] + X_0^+(y)\left[\frac{\pi^{\frac{1}{2}}}{4} - B_2\right] = \pi^{\frac{1}{2}} F_2 \tag{11}$$

with

$$F_1 = q_y + \frac{1}{\pi^{\frac{1}{2}}}\psi_w A_1\,, \quad F_2 = -q_y(y\lambda) + C - \frac{1}{\pi^{\frac{1}{2}}}\psi_w A_2 \tag{12a, b}$$

and

$$A_n = J_n\{\lambda(y-h)\}\,, \quad B_n = J_n\{\lambda y\}\,. \tag{12c}$$

Before determining the unknown functions $X_0^\pm(y)$, we first of all determine the constants q_y and C which occur in equations (10) and (11).

In this procedure the boundary conditions are automatically satisfied and q_y and C are determined as eigenvalues and $X_0^\pm(y)$ as eigenfunctions. Putting $y=0$ and $y=h$, we arrive at the following results:

★ Bhatnagar and Srivastava [1] did not include the factor $(T_0/T_h)^{\frac{1}{2}}$ in the boundary condition at the upper plate. This has been incorporated by Srivastava and Saraf [6].

$$q_y = -\frac{\left[\frac{1}{\pi^{\frac{1}{2}}}\psi_w\left(\frac{\pi^{\frac{1}{2}}}{4}a_1-\frac{1}{2}a_2\right)+b_1\left(\frac{\pi^{\frac{1}{2}}}{4}-a_2\right)+b_2(\frac{1}{2}-a_1)\right]}{2\left(\frac{\pi^{\frac{1}{2}}}{4}-a_2\right)+\alpha(\frac{1}{2}-a_1)}, \tag{13}$$

$$C = -\frac{1}{\pi^{\frac{1}{2}}}\psi_w\left[\frac{\left(\frac{\pi^{\frac{1}{2}}}{4}-a_2\right)\left\{\left(\frac{\pi^{\frac{1}{2}}}{4}-a_2\right)+\alpha(\frac{1}{2}-a_1)\right\}}{(\frac{1}{2}-a_1)\left\{2\left(\frac{\pi^{\frac{1}{2}}}{4}-a_2\right)+\alpha(\frac{1}{2}-a_1)\right\}}-\frac{\left\{\frac{\pi^{\frac{1}{2}}}{4}-b_1a_2-a_1b_2\right\}}{(\frac{1}{2}-a_1)}\right] \tag{14}$$

where

$$a_n = J_n(\alpha), \quad b_n = J_n(0) \tag{15}$$

and

$$\alpha = \lambda h \text{ (the inverse of the Knudsen number)}.$$

Solving the eigenfunctions $X_0^{\pm}(y)$ from equations (10) and (11), we have

$$X_0^+(y) = \frac{\pi^{\frac{1}{2}}\left[F_1\left(\frac{\pi^{\frac{1}{2}}}{4}-A_2\right)+F_2(\frac{1}{2}-A_1)\right]}{(\frac{1}{2}-B_1)\left(\frac{\pi^{\frac{1}{2}}}{4}-A_2\right)+\left(\frac{\pi^{\frac{1}{2}}}{4}-B_2\right)(\frac{1}{2}-A_1)} \tag{16}$$

and

$$X_0^-(y) = \frac{\pi^{\frac{1}{2}}\left[F_2(\frac{1}{2}-B_1)-F_1\left(\frac{\pi^{\frac{1}{2}}}{4}-B_2\right)\right]}{(\frac{1}{2}-B_1)\left(\frac{\pi^{\frac{1}{2}}}{4}-A_2\right)+\left(\frac{\pi^{\frac{1}{2}}}{4}-B_2\right)(\frac{1}{2}-A_1)}$$

where F_1 and F_2 are given in equation (12). Now the problem is completely solved and we will give only the heat flux vector and temperature jumps.

The dimensionless heat flux vector is

$$\frac{q_y}{q_{kn}} = \frac{\left(\frac{\pi^{\frac{1}{2}}}{4}a_1-\frac{1}{2}a_2\right)+b_1\left(\frac{\pi^{\frac{1}{2}}}{4}-a_2\right)+b_2(\frac{1}{2}-a_1)}{\left(\frac{\pi^{\frac{1}{2}}}{4}-a_2\right)+\frac{\alpha}{2}(\frac{1}{2}-a_1)} \tag{18}$$

where q_{kn} is the heat flux vector in the free molecular limit.

The temperature τ is given as

$$\tau(y) = \frac{2}{3\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}\psi(v_y, y)\exp(-v_y^2)dv_y$$

$$= \frac{2}{3\pi^{\frac{1}{2}}}\left[\psi_w J_0\{\lambda(y-h)\}+X_0^-(y)\left(\frac{\pi^{\frac{1}{2}}}{2}-J_0\{\lambda(y-h)\}\right)+X_0^+(y)\left(\frac{\pi^{\frac{1}{2}}}{2}-J_0\{\lambda y\}\right)\right] \tag{19}$$

From (19) the temperature jumps are

$$\frac{T(0)-T_0}{T_0} = \frac{2}{3\pi^{\frac{1}{2}}}\left[\psi_w J_0(\alpha)+X_0^-(0)\left\{\frac{\pi^{\frac{1}{2}}}{2}-J_0(\alpha)\right\}+X_0^+(0)\left\{\frac{\pi^{\frac{1}{2}}}{2}-J_0(0)\right\}\right], \tag{20}$$

$$\frac{T(h)-T_h}{T_0} = \left[1-\frac{T_h}{T_0}+\frac{2}{3\pi^{\frac{1}{2}}}\left\{\psi_w J_0(0)+X_0^-(h)\left\{\frac{\pi^{\frac{1}{2}}}{2}-J_0(0)\right\}+X_0^+(h)\left\{\frac{\pi^{\frac{1}{2}}}{2}-J_0(\alpha)\right\}\right\}\right] \tag{21}$$

where $X_0^{\pm}(0)$ and $X_0^{\pm}(h)$ can be obtained from equations (16) and (17) and ψ_w is defined by equation (2). The tables for the functions $J_n(\alpha)$ were obtained by Huang [5] by using Gauss–

Hermite quadrature. Fig. 1 shows the variation of heat flux vector with inverse Knudsen number and in Fig. 2 we have plotted the temperature profile $T(y)/T_0$ between the two plates against y for $\alpha=10$, $T_h/T_0=1.4$ and $U=0.3$. Although the approach is slightly cumbersome, our plots show fairly good agreement with the results obtained by Bhatnagar and Srivastava [1] and Bassinani *et al.* [7].

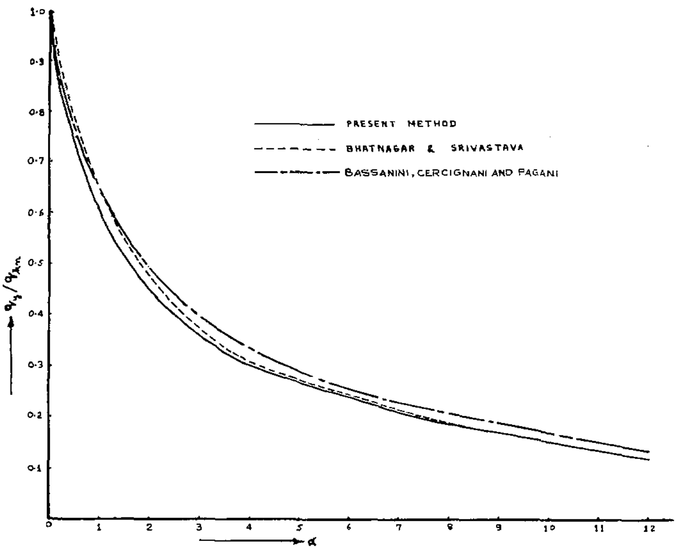

Figure 1. The heat flow *vs.* inverse Knudsen number.

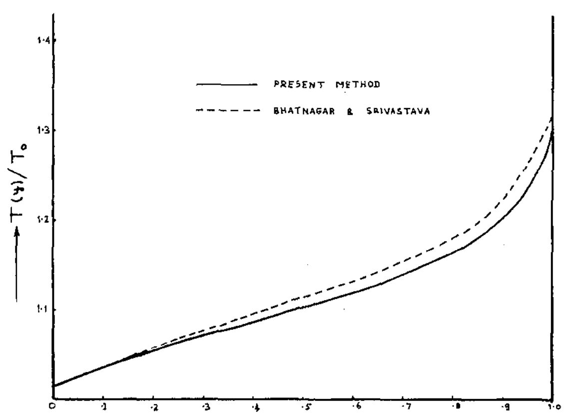

Figure 2. The temperature profile between the plates.

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